FACETS OF THE m-GENERALIZED CLUSTER COMPLEX AND REGIONS IN THE m-EXTENDED CATALAN ARRANGEMENT OF TYPE A_n

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ABSTRACT. In this paper we present a bijection ω_n between two well known families of Catalan objects: the set of facets of the m-generalized cluster complex $\Delta^m(A_n)$ and the set of dominant regions in the m-Catalan arrangement $\mathrm{Cat}^m(A_n)$, where $m \in \mathbb{N}_{>0}$. In particular, ω_n bijects the facets containing the negative simple root $-\alpha$ to dominant regions having the hyperplane $\{v \in V \mid \langle v, \alpha \rangle = m\}$ as separating wall. As a result, ω_n restricts to a bijection between the set of facets of the positive part of $\Delta^m(A_n)$ and the set of bounded dominant regions in $\mathrm{Cat}^m(A_n)$. The map ω_n is a composition of two bijections in which integer partitions in an m-staircase shape come into play.

1. INTRODUCTION

Fomin and Zelevinsky defined the *cluster complex* $\Delta(\Phi)$, a pure simplicial complex associated to every finite root system Φ [12], as a tool for the study of finite cluster algebras [11]. The ground set of $\Delta(\Phi)$ is the set of *almost positive roots*, $\Phi_{\geq -1}$, which consists of one copy of each negative and positive simple root. Fomin and Reading [10] generalized it to the *m-cluster complex* $\Delta^m(\Phi)$, a pure simplicial complex defined for any finite (possibly reducible) Coxeter group and nonnegative integer m. The ground set of $\Delta^m(\Phi)$ is the set of *colored almost positive roots*, $\Phi^m_{\geq -1}$, which consists of one copy of each negative simple root together with m copies of the each positive root. The complex $\Delta^m(\Phi)$ coincides with $\Delta(\Phi)$ when $m = 1$. The generalized cluster complex contains a natural subcomplex, called the positive part of $\Delta^m(\Phi)$ and denoted by $\Delta^m_+(\Phi)$. It consists of faces of $\Delta^m(\Phi)$ that do not contain negative simple roots.

When Φ is irreducible, the number of facets of $\Delta^m(\Phi)$ and $\Delta^m_+(\Phi)$ is equal to the *m-Catalan number* $N(\Phi, m) = \prod_{i=1}^n \frac{e_i + mh + 1}{e_i + 1}$ and *positive m-Catalan number* $N_+(\Phi, m) = \prod_{i=1}^n \frac{e_i + mh - 1}{e_i + 1}$ respectively, where n is the rank, h is the Coxeter number and e_i are the exponents of Φ. In the special case where $\Phi = A_{n-1}$, we obtain the well-known *Fuss-Catalan numbers*

$$N(\Phi, m) = \frac{1}{mn + 1} \binom{(m+1)n}{n}.$$

The Fuss-Catalan numbers count an enormous number of combinatorial objects. For instance, they count the number of m-Dyck (or m-ballot) paths of size n. These are lattice paths from $(0, 0)$ to (mn, n) using north steps $(0, 1)$ and east steps $(1, 0)$ which do not go below the line $y = \frac{1}{m}x$. Equivalently, they count the number of integer partitions whose Young diagram is contained in the m-staircase shape defined by the partition $(m(n-1), m(n-2), \ldots, m)$. In this paper we use such integer partitions as a way to encode the facets of the generalized cluster complex $\Delta^m(A_{n-1})$, as well as the dominant regions of the extended Catalan arrangement of type A_{n-1}.

The dominant regions of type A_n constitute the second object of our study. We now give a brief description of the dominant regions in the general case of a finite crystallographic root system Φ. For more information on this topic, we refer the reader to [2–4]. Let Φ be a finite crystallographic root system with set of positive roots $\Phi_{>0}$ and let V be the Euclidean space spanned by the set $\Phi_{>0}$ with inner product $\langle \cdot, \cdot \rangle$. The *$m$-Catalan arrangement* $\mathrm{Cat}^m(\Phi)$ is the collection of hyperplanes $\{H_{\alpha, k} \mid \alpha \in \Phi_{>0}, 0 \leq k \leq m\}$, where $H_{\alpha, k} = \{v \in V \mid \langle v, \alpha \rangle = k\}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. The hyperplanes of $\mathrm{Cat}^m(\Phi)$ dissect V into *regions*. The *dominant chamber* of V is the intersection $\bigcap_{\alpha \in \Phi_{>0}} \{v \in V \mid \langle v, \alpha \rangle \geq 0\}$, and is also referred to as the *fundamental chamber* in the literature. Every region contained in the dominant chamber is called a *dominant region*. Note that the dominant regions in the m-Catalan arrangement are the same as the dominant regions in the

m-Shi arrangement [21, 23]. The *bounded* regions are those which only contain $v \in V$ satisfying $0 < \langle v, \alpha \rangle < m$ for all simple roots α of Φ. When Φ is an irreducible crystallographic root system, the number of dominant and bounded dominant regions in $\mathrm{Cat}^m(\Phi)$ is again $N(\Phi, m)$ and $N_+(\Phi, m)$ respectively.

Summarizing, for every finite crystallographic root system we have the same number of facets of $\Delta^m(\Phi)$ as we have dominant regions in $\mathrm{Cat}^m(\Phi)$. Moreover, the number of facets of the positive cluster complex $\Delta_+^m(\Phi)$ is equal to the number of bounded dominant regions in the m-Catalan arrangement $\mathrm{Cat}^m(\Phi)$. We seek a bijection to explain these coincidences. For $m = 1$ there exist bijections between non-crossing partitions and cluster complexes [17] and between non-crossing partitions and dominant regions in $\mathrm{Cat}^1(\Phi)$ [1, 7, 16]. However, for $m \geq 2$ few results that relate the above objects are known [6, 15, 24]. To the best of our knowledge, there exists no bijection between dominant regions in $\mathrm{Cat}^m(\Phi)$ and facets of the cluster complex $\Delta^m(\Phi)$.

In this paper we close this gap in the case where $\Phi = A_n$. Before we state our main result we need to fix some notation. Let $\mathcal{D}^m(A_n)$ (or $\mathcal{D}_n^m$ for short) be the set of facets of the m-generalized cluster complex $\Delta^m(A_n)$ and let $\mathcal{R}^m(A_n)$ (or $\mathcal{R}_n^m$ for short) be the set of dominant regions in the m-Catalan arrangement $\mathrm{Cat}^m(A_n)$. We recall that a *wall* of a region R is a hyperplane in $\mathrm{Cat}^m(\Phi)$ which supports a facet of R. We say that H is a *separating wall* of R, if the region R and the origin lie in different half-spaces relative to H. Finally, for $n \in \mathbb{N}$ we set $[n] = \{1, 2, \ldots, n\}$. Here we give a combinatorial proof of the following theorem.

Theorem 1.1. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots of type A_n. There exists a bijection $\omega_n : \mathcal{D}_n^m \to \mathcal{R}_n^m$ with the property that the negative simple root $-\alpha_i$ is contained in the facet $D \in \mathcal{D}_n^m$ for $i \in [n]$, if and only if the hyperplane $H_{\alpha_i, m}$ is a separating wall of the region $\omega_n(D)$.*

We remark that the map ω_n of Theorem 1.1 restricts to a bijection between facets of $\Delta_+^m(A_n)$ and bounded regions in $\mathrm{Cat}^m(A_n)$. Even further, ω_n satisfies a certain refinement of this; it gives a combinatorial proof of the following observation, in case $\Phi = A_n$.

Proposition 1.2. *Let Φ be a finite crystallographic root system and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be its set of simple roots. For any $J \subseteq [n]$, the number of facets of $\Delta^m(\Phi)$ containing exactly the negative simple roots $-\alpha_i$ with $i \in J$, is equal to the number of dominant regions in $\mathrm{Cat}^m(\Phi)$ with simple separating walls $H_{\alpha_i, m}$ with $i \in J$.*

In order to construct the bijection ω_n, we associate dominant regions of the m-Catalan arrangement $\mathrm{Cat}^m(A_n)$ to facets of the m-generalized cluster complex $\Delta^m(A_n)$ via certain integer partitions. More precisely, as mentioned above, we use as intermediaries integer partitions whose Young diagram fits inside the diagram of the partition $(mn, m(n-1), \ldots, m)$ and we call them *type-A* partitions of size (n, m). We denote the set of such partitions by $\mathcal{P}^m(A_n)$ (or $\mathcal{P}_n^m$ for short). Partitions in $\mathcal{P}_n^m$ can also be viewed as lattice paths from $(0, 0)$ to (mn, n) using north and east steps which stay above the line $y = \frac{1}{m}x - 1$, or equivalently m-Dyck, or m-ballot paths of size $n + 1$. Our main idea is to biject both facets of the cluster complex and dominant Catalan regions to type-A partitions. A facet containing the negative simple root $-\alpha_i$ becomes a partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ whose i-th part has maximum size, that is $\lambda_i = (n - i + 1)m$. On the other hand, a dominant region with simple separating wall $H_{\alpha_i, m}$ also bijects to a partition whose i-th part has maximum size. We remark that a maximum-sized part in a type-A partition corresponds to a point where the lattice path touches a line of the form $y = x/m + t$, where $t \in \mathbb{R}$, or where the m-Dyck path touches the line $y = \frac{1}{m}x$. Touch points (also called contacts) appear often in the literature. For instance, in [5], touch points are used to derive a recursion to count the number of intervals in a generalization of the m-Tamari lattice and in [14] sorting 1-Dyck paths by touch points leads to a refinement of the shuffle conjecture. In this paper we prove combinatorially the following theorems, which eventually lead to the construction of ω_n.

Theorem 1.3. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots of type A_n. There exists a bijection $\varphi_n : \mathcal{R}_n^m \to \mathcal{P}_n^m$ with the property that the hyperplane $H_{\alpha_i, m}$ is a separating wall of the region $R \in \mathcal{R}_n^m$ for $i \in [n]$, if and only if the i-th part of the partition $\varphi_n(R)$ is equal to $(n - i + 1)m$.*

Theorem 1.4. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots of type A_n. There exists a bijection $\psi_n : \mathcal{D}_n^m \to \mathcal{P}_n^m$ with the property that the negative simple root $-\alpha_i$ is contained in the facet $D \in \mathcal{D}_n^m$ for $i \in [n]$, if and only if the i-th part of the partition $\psi_n(D)$ is equal to $(n - i + 1)$.*

For our bijections we use a realization of the facets of the cluster complex in terms of polygon dissections given by S. Fomin and N. Reading [10]. Also, we realize the regions in terms of certain tableaux which we call *type-A Shi tableaux*.

This paper is structured as follows. In Section 2 we provide the necessary background, fix notation and prove Proposition 1.2, which constitutes the motivation for our work. Since the realization of the dominant regions in Cat_n^m as Shi tableaux will be crucial for our proofs, in Section 3 we describe these tableaux in detail. In Section 4 we prove Theorem 1.3. Theorem 1.4 is proved in Section 5. We complete this paper with Section 6, where we prove Theorem 1.1 by composing the bijections of Theorems 1.3 and 1.4. We also include an application of Theorem 1.4, where we rediscover the positive Catalan numbers of type A_n bijectively. We therefore answer [4, Remark 2] in the case where k is the rank of the root system, and $\Phi = A_n$.

2. Preliminaries

2.1. **Root system of type** A_n. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$. In type A_n, a standard choice of positive and simple roots are respectively, the sets $\{\alpha_{ij} \mid 1 \leq i \leq j \leq n\}$ and $\{\alpha_i \mid 1 \leq i \leq n\}$, where $\alpha_{ij} := \varepsilon_i - \varepsilon_{j+1}$ and $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, for every $1 \leq i \leq j \leq n$. The positive roots can be written in terms of simple roots as

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \text{ for every } 1 \leq i \leq j \leq n.$$

For instance, the positive roots of A_2 are the vectors: $\alpha_1 = (1, -1, 0), \alpha_2 = (0, 1, -1)$ and $\alpha_{12} = (1, 0, -1)$.

2.2. **Cluster complexes and polygon dissections.** The cluster complex $\Delta^m(A_n)$ is a pure simplicial complex of dimension $n - 1$ on the ground set of *colored almost positive roots* which consists of m "colored" copies of each positive root and one copy of each negative simple root. This simplicial complex can be described in terms of *polygon dissections*. We refer the reader to [10] for background on the generalized cluster complex.

Let P be a polygon with $m(n+1)+2$ vertices labeled from 0 up to $m(n+1)+1$ in counterclockwise order. An m-*diagonal* in P is a diagonal dissecting P into a pair of polygons, where for each polygon the number of vertices is congruent to 2 modulo m. A collection of n such m-diagonals is a $(m + 2)$-*angulations* of P. We call such a $(m + 2)$-angulation a *type-A (polygon) dissection* of size (n, m), or m-*dissection* for short.

The faces of $\Delta^m(A_n)$ of dimension k are in bijection with dissections of P having $k+1$ pairwise non crossing m-diagonals. The facets of $\Delta^m(A_n)$ correspond to m-dissections of P. For $1 \leq i \leq \frac{n+1}{2}$, the negative simple root $-\alpha_{2i-1}$ is identified with the diagonal of P connecting the vertex $(i-1)m$ to the vertex $(n+1-i)m+1$. For $1 \leq i \leq \frac{n}{2}$, we identify the negative simple root $-\alpha_{2i}$ with the diagonal of P connecting the vertex im to the vertex $(n + 1 - i)m + 1$. We notice that the negative simple roots form a "snake" in P (see Figure 1).
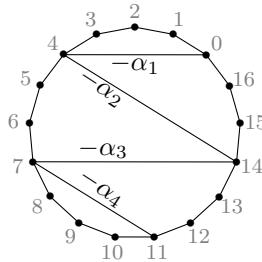

FIGURE 1. The snake formed by the negative simple roots for $\Delta^3(A_4)$.

For each pair (i, j) with $1 \leq i \leq j \leq n+1$ there are exactly n many m-diagonals that intersect (in the interior) the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and no other diagonals in the snake. The diagonal corresponding to the colored positive root α_{ij}^k is uniquely determined as the k-th in clockwise order of the exactly n m-diagonals which intersect $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$. Under this identification, every m-diagonal in P corresponds to an almost positive root in $\Delta^m(A_n)$ [10, Section 5.1]. In this paper we denote by $\mathcal{D}^m(A_n)$ (or $\mathcal{D}_n^m$ for short) the set of m-dissections of an $(m(n+1)+2)$-gon.

Definition 2.1. *We say that $D \in \mathcal{D}_n^m$ contains the negative simple root $-\alpha_i$, for some $i \in [n]$, if the diagonal corresponding to $-\alpha_i$ is contained in D.*

2.3. **Integer partitions.** An *integer partition* is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative integers, called parts. We identify a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with its *Young diagram*, the left-justified array of boxes with λ_i boxes in row i. The box in row i, column j, has coordinates (i, j). In what follows we describe the partitions we are interested in. Let n and m be positive integers. We denote by $\mathcal{P}^m(A_n)$ (or $\mathcal{P}_n^m$ for short) the set of partitions whose Young diagram fits into an m-staircase shape of size n (see Figure 2):

$$\mathcal{P}_n^m = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \mid 0 \leq \lambda_i \leq m(n-i+1)\}.$$
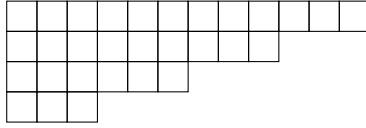

FIGURE 2. A 3-staircase shape of size 4.

The number of such partitions is $\frac{1}{m(n+1)+1}\binom{(m+1)(n+1)}{n+1}$. Throughout this paper we will refer to partitions in $\mathcal{P}_n^m$ as (n, m)-*type A* partitions (or just *type-A* partitions, when there is no ambiguity).

2.4. **Dominant regions in the m-Catalan arrangement and Shi tableaux.** Shi [20] and others arrange the positive roots in diagrams which encode the poset structure of the positive roots. We use this idea to store coordinates describing the location of dominant regions of the Catalan arrangement. We call the diagrams *Shi tableaux*.

Let R be a dominant region in $\mathrm{Cat}^m(\Phi)$. By definition of the m-Catalan arrangement, for each $\alpha \in \Phi_{>0}$ there exists an integer $0 \leq k_\alpha \leq m$ such that for all $x \in R$ we have $k_\alpha \leq \langle \alpha, x \rangle \leq k_\alpha'$, where

$$k_\alpha' = \begin{cases} k_\alpha + 1 & \text{if } 0 \leq k_\alpha \leq m-1, \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 2.2. *We define the* Shi tableau *for a dominant region R in $\mathrm{Cat}^m(\Phi)$ to be the set of positive integers $\{k_\alpha \mid \alpha \in \Phi_{>0}\}$.*

Each Shi tableau is attached to a dominant region. More precisely, for each $\alpha \in \Phi_{>0}$, the coordinate k_α is the number of integer translations of the hyperplane H_α which separate the region from the origin.

2.5. **Reducible crystallographic root systems and the proof of Proposition 1.2.** The definition of the cluster complex $\Delta^m(\Phi)$ as well as that of the m-Catalan arrangement can be generalized to reducible crystallographic root systems as follows. If $\Phi = \Phi_1 \times \cdots \times \Phi_\ell$ is the factorization of Φ into irreducible root systems, then $\Delta^m(\Phi)$ is defined as the simplicial join of $\Delta^m(\Phi_i)$ with $1 \leq i \leq \ell$ [10, Section 3], thus $N(\Phi, m) = \prod_{i=1}^\ell N(\Phi_i, m)$. Moreover, the set of facets of $\Delta^m(\Phi)$ containing no negative simple roots is equal to the product $\prod_{i=1}^\ell N_+(\Phi_i, m)$ [10, Corollary 12.1]. On the other hand, the m-Catalan arrangement $\mathrm{Cat}^m(\Phi)$ generalizes straightforwardly in the case where Φ is reducible. Clearly, the number of regions and bounded regions is equal to $\prod_{i=1}^\ell N(\Phi_i, m)$ and $\prod_{i=1}^\ell N_+(\Phi_i, m)$ respectively. In what follows, we write Φ_I instead of Φ,

where I is an index set in bijection with the set Π of simple roots. For $J \subseteq I$ we will denote by Φ_J the parabolic root system with simple roots $\{\alpha_i \mid i \in J\}$. Note that Φ_J may be reducible, even in the case where Φ_I is irreducible.

Proof of Proposition 1.2. From [10, Theorem 3.7] we deduce that the number of facets of $\Delta^m(\Phi_I)$ having exactly the negative simple roots $-\alpha_i, i \in J$ is equal to the number of facets of $\Delta^m(\Phi_{I \setminus J})$ having no negative simple roots. From the proof of [4, Lemma 5.3] we deduce that the number of dominant regions in $\text{Cat}^m(\Phi_I)$ with simple separating walls $H_{\alpha_i}, i \in J$ is equal to the number of bounded dominant regions in $\text{Cat}^m(\Phi_{I \setminus J})$. The result then follows, since the number of facets of $\Delta^m(\Phi_{I \setminus J})$ having no negative simple roots is equal to the number of bounded dominant regions in $\text{Cat}^m(\Phi_{I \setminus J})$. $\square$

Notation. For $n \in \mathbb{N}$ we denote by $[n]$ and $[n]_{\geq 0}$ the sets $\{1, 2, \ldots, n\}$ and $\{0, 1, 2, \ldots, n\}$, respectively.

3. SHI TABLEAUX

In this section we describe a way to represent the Shi tableaux for type-A dominant regions. Let R be a dominant region in $\mathcal{R}_n^m$ and let T be a staircase Young diagram of size $(n, n-1, \ldots, 1)$. We arrange the coordinates $k_{\alpha_{ij}}$ corresponding to R in the diagram T so that for each $1 \leq i \leq j \leq n$, the integer $k_{\alpha_{i,j}}$ is placed in the box $(i, n-j+1)$. To simplify the notation, we write $k_{i,j}$ instead of $k_{\alpha_{ij}}$ (see Figure 3).

$k_{1,4}$	$k_{1,3}$	$k_{1,2}$	$k_{1,1}$
$k_{2,4}$	k_{23}	$k_{2,2}$	
$k_{3,4}$	$k_{3,3}$		
$k_{4,4}$			

FIGURE 3. The distribution of the coordinates in a 4-staircase diagram for type A_4.

Throughout this paper, we identify each dominant regions of an m-Catalan arrangement $\mathcal{R}_n^m$ with its Shi tableau.

The entries of a Shi tableau satisfy certain conditions [3]. In order to describe them explicitly, we need to fix some notation. Let x be a box in a Young diagram T. A *hook* h *of length ℓ on a box* x of T is an array of ℓ contiguous boxes lying to the right (and same row) or below (and same column) of x including the box x itself. The box x is called *the corner* of the hook. For a Young diagram whose boxes are filled with numbers, we denote by $\mathsf{e}(\mathsf{h})$ the sum of the numbers at the endpoints of the hook h. The following characterization for the entries of a Shi tableau can be deduced from results in [3, 19]. A more detailed proof can be found in [9, Lemma 2.5].

Proposition 3.1. *Let $T = \{k_{i,j} \mid 1 \leq i \leq j \leq n\}$ be a set of integers in $[m]_0$ arranged in a staircase diagram of size $(n, n-1, \ldots, 1)$ so that $k_{i,j}$ is placed in the box $(i, n-j+1)$. Then, T is a type-A Shi tableau of some region $R \in \mathcal{R}_n^m$ if and only if for every $1 \leq i < j \leq n$ and any hook $\mathsf{h}_{k_{i,j}}$ on $k_{i,j}$ of length $j - i + 2$ we have*

$$(3.1) \qquad k_{i,j} = \begin{cases} \mathsf{e}(\mathsf{h}_{k_{i,j}}) + \delta, & \text{if } \mathsf{e}(\mathsf{h}_{k_{i,j}}) < m \\ m & \text{otherwise,} \end{cases}$$

where $\delta \in \{0, 1\}$. *Equivalently, T is a type-A Shi tableau of some region $R \in \mathcal{R}_n^m$ if and only if for every $1 \leq i < j \leq n$ and $i \leq \ell < j$ we have*

$$(3.2) \qquad k_{i,j} = \begin{cases} k_{i\ell} + k_{\ell+1,j} + \delta, & \text{if } k_{i\ell} + k_{\ell+1,j} < m \\ m & \text{otherwise,} \end{cases}$$

where $\delta \in \{0, 1\}$.

We say that the entry (or corner) $k_{i,j}$ of T satisfies the Shi conditions, if Equation (3.2) holds for all $i \leq \ell < j$. Furthermore, if Equation (3.2) holds for some fixed ℓ, then we say that the triplet $\{k_{i,j}, k_{i,\ell}, k_{\ell+1,j}\}$ satisfies the Shi conditions.

Definition 3.2. *Let T be a staircase Young diagram of size $(n, n-1, \ldots, 1)$ and let $k_{i,j}$ be the entry of the box $(i, n - j + 1)$. If the Shi conditions hold for every $1 \leq i \leq j \leq n$, we say that T is a type-A Shi tableau of size (n, m).*

Figure 4 illustrates the dominant regions of the hyperplane arrangements $\mathrm{Cat}^3(A_2)$ together with their Shi tableaux.
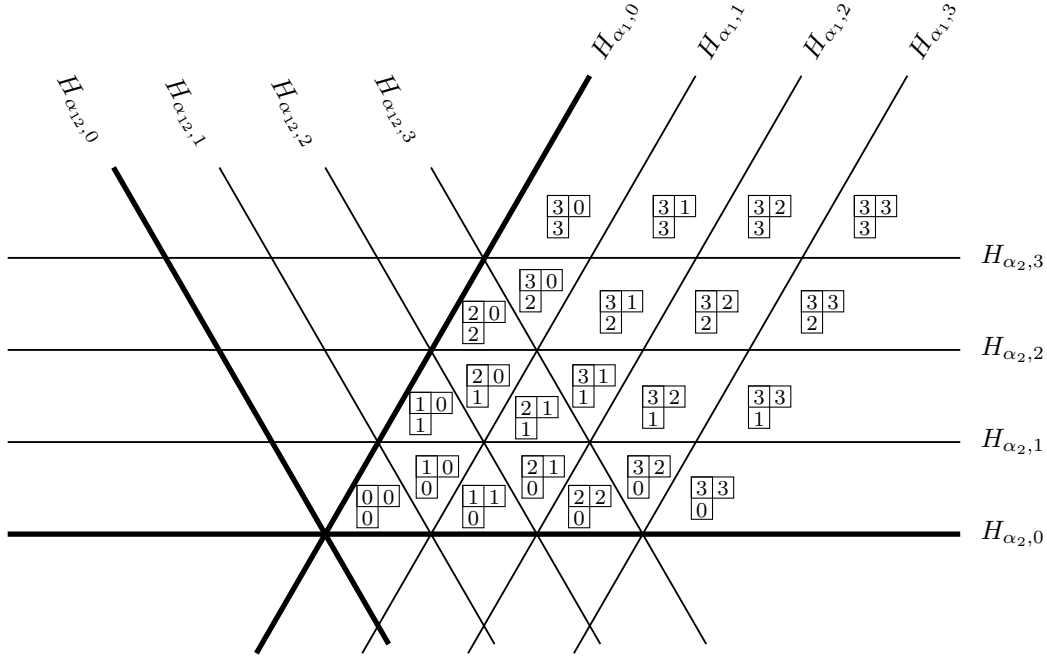

FIGURE 4. The hyperplane arrangement $\mathrm{Cat}^3(A_2)$ together with the Shi tableaux that correspond to each dominant region.

4. FROM DOMINANT REGIONS $\mathcal{R}_n^m$ TO PARTITIONS IN $\mathcal{P}_n^m$

In this section we prove Theorem 1.3 combinatorially. In particular, we provide a bijection φ_n between the set $\mathcal{R}_n^m$ of dominant regions and the set $\mathcal{P}_n^m$ of type-A partitions. More precisely, for a simple root α of type A_n, the bijection φ_n characterizes the dominant regions of $\mathcal{R}_n^m$ which are separated from the origin by the hyperplane $H_{\alpha,m}$, it terms of partitions in $\mathcal{P}_n^m$. The formula for φ_n is given in Theorem 4.1 and its inverse is explicitly described in Theorem 4.2. Before we proceed to the proofs, a few comments on the history of this bijection are in order. In [22, 23], Stanley defined a bijection between regions in the extended Shi arrangement and generalized parking functions. In both papers, the bijection is recursively constructed. When restricted to the dominant chamber, the parking functions in the image can be seen as partitions and ours agrees with it. Richards [18] explictly defined a bijection from dominant regions to partitions and again,

ours agrees with it. He proved it was an injection and used the sizes of the sets involved to prove it was a bijection. In Theorem 4.2, we give a formula for the inverse of the function in [18], thereby showing directly that we have a bijection.

Let $\mathcal{T}_n^m$ denote the set of staircase tableaux of size $(n, n-1, \ldots, 1)$, whose entries are positive integers between 0 and m, and let $k_{i,j}$ be the entry of the box $(i, n-j+1)$. Note that $\mathcal{R}_n^m$ can be considered as a proper subset of $\mathcal{T}_n^m$. We define $\phi_n : \mathcal{T}_n^m \to \mathcal{P}_n^m$ to be the map which sends each tableau $T = \{k_{i,j} \mid 1 \le i \le j \le n\} \in \mathcal{T}_n^m$ to the partition $\lambda_T = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with parts $\lambda_i = \sum_{j=i}^{n} k_{i,j}$. Clearly ϕ_n is a surjection. Let φ_n denote the restriction of the map ϕ_n to the set $\mathcal{R}_n^m$.

Theorem 4.1. *The map φ_n is a bijection between the sets $\mathcal{R}_n^m$ and $\mathcal{P}_n^m$.*

It is immediate from the definition of the Shi tableaux that the map φ_n is well defined. In order to show that φ_n is a bijection we will first construct its inverse. The first step towards this direction constitutes the next result.

Theorem 4.2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}_n^m$. For every $1 \le i \le j \le n$ define*

$$(4.1) \qquad k_{i,j}(\lambda) = \min \left\{ m, \left\lceil \frac{\lambda_i - \sum_{\ell=j+1}^{n} k_{i,\ell}(\lambda) + \sum_{\ell=i+1}^{j} k_{\ell,j}(\lambda)}{j - i + 1} \right\rceil \right\}.$$

The map $\bar{\varphi}_n : \mathcal{P}_n^m \to \mathcal{T}_n^m$, which sends each λ to the set $\{k_{i,j}(\lambda) \mid 1 \le i \le j \le n\}$, is an injection.

In the remainder of this section, when there is no ambiguity we will write $k_{i,j}$ instead of $k_{i,j}(\lambda)$. For the proof of Theorem 4.2 we need the following lemma.

Lemma 4.3. *Let $\lambda \in \mathcal{P}_n^m$, and let $k_{i,j}(\lambda) = k_{i,j}$ be defined as in Equation (4.1). Then, $k_{i,j} \ge k_{i,j-1}$ for all $1 \le i < j \le n$.*

Proof. We proceed by induction on n. Let $n = 2$. We will show that $k_{1,2} \ge k_{1,1}$. If $k_{1,2} = m$ our claim is trivially true, so we may suppose that $k_{1,2} < m$. Since $k_{1,1} = \min\{m, \lambda_1 - k_{1,2}\}$, it is enough to show that $k_{1,2} \ge \lambda_1 - k_{1,2}$, or equivalently that $2k_{1,2} \ge \lambda_1$. We have $2k_{1,2} = 2\left\lceil \frac{\lambda_1 + k_{2,2}}{2} \right\rceil \ge \lambda_1 + k_{2,2}$. Now, from Equation (4.1) we deduce that $\lambda_2 = k_{2,2}$, and therefore $2k_{1,2} \ge \lambda_1 + \lambda_2 \ge \lambda_1$ and we are done.

By applying induction, we may assume that $k_{l,n} \ge k_{l,n-1}$ for every $2 \le \ell \le n-1$, and thus it remains to show that $k_{1,n} \ge k_{1,n-1}$ as well. As before, the result is trivial when $k_{1,n} = m$, so we may suppose that $k_{1,n} < m$. In this case $k_{1,n} = \left\lceil \frac{\lambda_1 + \sum_{\ell=2}^{n} k_{\ell,n}}{n} \right\rceil = \frac{\lambda_1 + \sum_{\ell=2}^{n} k_{\ell,n} + v}{n}$ for some $v \in [n-1]_{\ge 0}$.

We first show that

$$\frac{\lambda_1 + \sum_{\ell=2}^{n} k_{\ell,n}}{n} \ge \frac{\lambda_1 - k_{1,n} + \sum_{\ell=2}^{n-1} k_{\ell,n-1}}{n-1}.$$

Indeed, the above inequality holds if and only if

$$(n-1)(\lambda_1 + \sum_{\ell=2}^{n} k_{\ell,n}) \geq n(\lambda_1 - k_{1,n} + \sum_{\ell=2}^{n-1} k_{\ell,n-1}) \Leftrightarrow$$

$$-\lambda_1 + (n-1)\sum_{\ell=2}^{n} k_{\ell,n} + nk_{1,n} - n\sum_{\ell=2}^{n-1} k_{\ell,n-1} \geq 0 \Leftrightarrow$$

$$(n-1)\sum_{\ell=2}^{n} k_{\ell,n} + \sum_{\ell=2}^{n} k_{\ell,n} + \upsilon - n\sum_{\ell=2}^{n-1} k_{\ell,n-1} \geq 0 \Leftrightarrow$$

$$n\left(\sum_{\ell=2}^{n} k_{\ell,n} - \sum_{\ell=2}^{n-1} k_{\ell,n-1}\right) + \upsilon \geq 0 \Leftrightarrow$$

$$n\sum_{\ell=2}^{n-1} (k_{\ell,n} - k_{\ell,n-1}) + nk_{n,n} + \upsilon \geq 0,$$

which holds, since $k_{n,n} = \lambda_n \geq 0$, $\upsilon \geq 0$ and (by induction) $k_{l,n} \geq k_{l,n-1}$ for every $2 \leq \ell \leq n-1$.
   Thus

$$\frac{\lambda_1 + \sum_{\ell=2}^{n} k_{\ell,n}}{n} \geq \frac{\lambda_1 - k_{1,n} + \sum_{\ell=2}^{n-1} k_{\ell,n-1}}{n-1},$$

and therefore

$$\left\lceil \frac{\lambda_1 + \sum_{\ell=2}^{n} k_{\ell,n}}{n} \right\rceil \geq \left\lceil \frac{\lambda_1 - k_{1,n} + \sum_{\ell=2}^{n-1} k_{\ell,n-1}}{n-1} \right\rceil,$$

or equivalently, since $k_{1,n} < m$,

$$k_{1,n} \geq \left\lceil \frac{\lambda_1 - k_{1,n} + \sum_{\ell=2}^{n-1} k_{\ell,n-1}}{n-1} \right\rceil.$$

Finally, since $k_{1,n-1} = \min\left\{m, \left\lceil \frac{\lambda_1 - k_{1,n} + \sum_{\ell=2}^{n-1} k_{\ell,n-1}}{n-1} \right\rceil\right\}$, we deduce that $k_{1,n} \geq k_{1,n-1}$. This completes the induction and the proof of the lemma. $\square$

Proof of Theorem 4.2. It suffices to show that if $\phi_n(\bar{\varphi}_n(\lambda)) := \mu = (\mu_1, \mu_2, \ldots, \mu_n)$, then $\mu = \lambda$. We fix some $1 \leq i \leq n$ and distinguish two cases: (1) $k_{i,i}(\lambda) < m$ and (2) $k_{i,i}(\lambda) = m$.

   (1) Let $k_{i,i}(\lambda) < m$. Then, Equation (4.1) implies that $k_{i,i}(\lambda) = \lambda_i - \sum_{\ell=i+1}^{n} k_{i,\ell}(\lambda)$, therefore,
$\lambda_i = \sum_{\ell=i}^{n} k_{i,\ell}(\lambda) = \mu_i$.
   (2) Let $k_{i,i}(\lambda) = m$. Then, it follows from Lemma 4.3 that $k_{i,\ell}(\lambda) = m$ for all $i < \ell \leq n$. Furthermore, since $k_{i,i}(\lambda) = m$, Equation (4.1) implies that $\lambda_i - \sum_{\ell=i+1}^{n} k_{i,\ell}(\lambda) \geq m$ or equivalently that $\lambda_i \geq m + \sum_{\ell=i+1}^{n} k_{i,\ell}(\lambda) = m(n-i+1)$. However, $\lambda \in \mathcal{P}_n^m$, thus $\lambda_i \leq m(n-i+1)$. We therefore conclude that $\lambda_i = m(n-i+1) = \sum_{\ell=i+1}^{n} k_{i,\ell}(\lambda) = \mu_i$.

   Hence $(\mu_1, \mu_2, \ldots, \mu_n) = \lambda$. From the above discussion we deduce that for every $\lambda \in \mathcal{P}_n^m$ we have $\phi_n(\bar{\varphi}_n(\lambda)) = \lambda$. This implies that $\bar{\varphi}_n$ is an injection. $\square$

Corollary 4.4. *Let $\lambda \in \mathcal{P}_n^m$, and let $k_{i,j}(\lambda) = k_{i,j}$ be defined as in Equation (4.1). Then, $\lambda_i = \sum_{j=i}^{n} k_{i,j}$, for every $i \in [n]$.*

Proof. The result follows immediately from Theorem 4.2. $\square$

   In view of Theorem 4.2 and Corollary 4.4, in order to prove Theorem 4.1, it suffices to show that $\bar{\varphi}_n(\mathcal{P}^m) = \mathcal{R}_n^m$ (see Lemma 4.7 below). For this proof we need the following two lemmas.

Lemma 4.5. *Let $\lambda \in \mathcal{P}_n^m$ with $\bar{\varphi}_n(\lambda) = \{k_{i,j} \mid 1 \leq i \leq j \leq n\}$. Consider the partitions*

(1) $\lambda^{(1)} = (\lambda_2, \lambda_3, \ldots, \lambda_n)$.
(2) $\lambda^{(2)}$, with parts $\lambda_i^{(2)} = \lambda_i - k_{i,n}$, for every $1 \leq i \leq n - 1$.
(3) $\lambda^{(3)}$, with parts $\lambda_i^{(3)} = \lambda_i^{(2)} - k_{i,n-1}$, for every $1 \leq i \leq n - 2$.
(4) $\lambda^{(4)}$, with parts $\lambda_i^{(4)} = \lambda_i - k_{i,n-1}$, for every $1 \leq i \leq n - 1$.

Then,

- *$\lambda^{(1)}, \lambda^{(2)}, \lambda^{(4)} \in \mathcal{P}_{n-1}^m$ and*
- *$\lambda^{(3)} \in \mathcal{P}_{n-2}^m$.*

Moreover, if

$$T^{(h)} = \{k_{i,j}^{(h)} \mid 1 \leq i \leq j \leq n - 1\} := \bar{\varphi}_{n-1}(\lambda^{(h)}), \text{ for } h \in \{1, 2, 4\}$$

and

$$T^{(3)} = \{k_{i,j}^{(3)} \mid 1 \leq i \leq j \leq n - 2\} := \bar{\varphi}_{n-2}(\lambda^{(3)}),$$

then the following relations hold:

$$(4.2) \qquad k_{i,j}^{(1)} = k_{i+1,j+1}, \text{ for every } 1 \leq i \leq j \leq n - 1,$$

$$(4.3) \qquad k_{i,j}^{(2)} = k_{i,j}, \text{ for every } 1 \leq i \leq j \leq n - 1,$$

$$(4.4) \qquad k_{i,j}^{(3)} = k_{i,j}, \text{ for every } 1 \leq i \leq j \leq n - 2,$$

$$(4.5) \qquad k_{i,j}^{(3)} = k_{i,j}^{(4)}, \text{ for every } 1 \leq i \leq j \leq n - 2,$$

and

$$(4.6) \qquad k_{i,j}^{(4)} = \begin{cases} k_{i,j+1}, & \text{if } j = n - 1 \\ k_{i,j}, & \text{if } j \leq n - 2, \end{cases} \quad \text{for every } 1 \leq i \leq j \leq n - 1.$$

Equations (4.2)-(4.6) explain the relation between each partition λ with tableau T and partition $\lambda^{(h)}$ with tableau $T^{(h)}$, for $h = 1, 2, 3, 4$. In particular:

(1) The partition $\lambda^{(1)}$ is obtained from λ by omitting its first part, while the tableau $T^{(1)} = \bar{\varphi}_{n-1}(\lambda^{(1)})$ is obtained from T by removing its top row.
(2) The partition $\lambda^{(2)}$ is obtained from λ by subtracting (for each i) the i-th entry of the leftmost column of T from λ_i, while the tableau $T^{(2)} = \bar{\varphi}_{n-1}(\lambda^{(2)})$ is obtained from T by removing its leftmost column.
(3) The partition $\lambda^{(3)}$ is obtained from λ by subtracting (for each i) the i-th entries of the two leftmost columns of T from λ_i, while the tableau $T^{(3)} = \bar{\varphi}_{n-2}(\lambda^{(3)})$ is obtained from T by removing its two leftmost columns.
(4) The partition $\lambda^{(4)}$ is obtained from λ by subtracting (for each i) the i-th entry of the second leftmost column of T from λ_i, while the tableau $T^{(4)} = \bar{\varphi}_{n-1}(\lambda^{(4)})$ is obtained from T by removing its second leftmost column as well as its bottom leftmost box.

Figure 5 illustrates the tableaux T and $T^{(h)}$ for $h \in 1, 2, 3, 4$.

Proof. We first show that $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(4)} \in \mathcal{P}_{n-1}^m$ and $\lambda^{(3)} \in \mathcal{P}_{n-2}^m$. The fact that $\lambda^{(1)} \in \mathcal{P}_{n-1}^m$ is immediate from its definition. For $\lambda^{(2)}$ recall that $k_{i,j} \in [m]_0$, for all $1 \leq i \leq j \leq n$, and also that $\lambda_i = \sum_{j=i}^{n} k_{i,j}$ for every $i \in [n]$ (Corollary 4.4). We therefore conclude that $\lambda_i^{(2)} = \lambda_i - k_{i,n} \leq m(n-i)$, for every $i \in [n]$, which shows that $\lambda^{(2)} \in \mathcal{P}_{n-1}^m$. Our claim for $\lambda^{(3)}$ and $\lambda^{(4)}$ is completely analogous.

FIGURE 5. The tableau T is displayed on the left followed by the tableaux of the partitions $\lambda^{(h)}$, for $h = 1, 2, 3, 4$, respectively.

Equations (4.2)-(4.5) follow directly from the definition of $k_{i,j}^{(h)}$. We focus on Equation (4.6). We first prove the argument for $j = n - 1$. Let $1 \leq i \leq n - 1$. Using Equations (4.3), (4.4) and (4.5) we get the following implications:

$$\lambda_i = k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-1} + k_{i,n} \Rightarrow$$
$$\lambda_i - k_{i,n-1} = k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-1} + k_{i,n} \Rightarrow$$
$$\lambda_j^{(4)} = k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-1} + k_{i,n} \Rightarrow$$
$$k_{i,i}^{(4)} + k_{i,i+1}^{(4)} + \cdots + k_{i,n-1}^{(4)} = k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-1} + k_{i,n} \Rightarrow$$
$$k_{i,i}^{(3)} + k_{i,i+1}^{(3)} + \cdots + k_{i,n-2}^{(3)} + k_{i,n-1}^{(4)} = k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-1} + k_{i,n} \Rightarrow$$
$$k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-2} + k_{i,n-1}^{(4)} = k_{i,i} + k_{i,i+1} + \cdots + k_{i,n-1} + k_{i,n} \Rightarrow$$
$$k_{i,n-i}^{(4)} = k_{i,n}.$$

For $j \leq n-2$ the result follows directly from Equations (4.4) and (4.5). Indeed, for every $i \in [n-2]$ we have $k_{i,j}^{(4)} = k_{i,j}^{(3)} = k_{i,j}$. $\hfill\square$

Lemma 4.3 states that the entries $k_{i,j}$ from Equation (4.1) decrease from left to right along the rows. The following lemma shows that an analogous property holds for the columns. More precisely, we show that the entries $k_{i,j}$ decrease from top to bottom.

Lemma 4.6. *Let $\lambda \in \mathcal{P}_n^m$ and $k_{i,j}(\lambda) = k_{i,j}$ defined as in Equation (4.1). Then, $k_{i,j} \geq k_{i+1,j}$ for all $1 \leq i < j \leq n$.*

Proof. We proceed again by induction on n. Let $n = 2$. We will show that $k_{1,2} \geq k_{2,2}$. We may assume that $k_{1,2} < m$ since otherwise the result is trivial. Then, $k_{1,2} = \left\lceil \frac{\lambda_1 + k_{2,2}}{2} \right\rceil \geq \left\lceil \frac{k_{1,1} + k_{1,1}}{2} \right\rceil = k_{1,1}$. By induction and using the tableaux $T^{(h)}$ of Lemma 4.5 the result is trivial. $\hfill\square$

Lemma 4.7. *Let $\bar{\varphi}_n$ be the map of Theorem 4.2. Then $\bar{\varphi}_n(\mathcal{P}_n^m) = \mathcal{R}_n^m$.*

Proof. We will first prove that $\bar{\varphi}_n(\mathcal{P}_n^m) \subseteq \mathcal{R}_n^m$. Let $\lambda \in \mathcal{P}_n^m$. We will show that $\bar{\varphi}_n(\lambda) = \{k_{i,j}(\lambda), \ 1 \leq i \leq j \leq n\}$ is an element of $\mathcal{R}_n^m$. Equivalently, we will show that for every $1 \leq i \leq j \leq n$, the numbers $k_{i,j}$ satisfy the Shi conditions.

We proceed by induction on n. The result is trivial for $n = 1$. We treat now the case $n = 2$. Suppose first that $k_{1,2} < m$. Then $k_{1,2} = \frac{\lambda_1 + k_{2,2} + \upsilon}{2}$, where $\upsilon \in \{0, 1\}$. Since $\lambda_1 = k_{1,2} + k_{2,2}$, we conclude that $k_{1,2} = k_{1,1} + k_{2,2} + \upsilon$ and we are done. Suppose now that $k_{1,2} = m$. Then $m \leq \frac{\lambda_1 + k_{2,2} + \upsilon}{2}$, where $\upsilon \in \{0, 1\}$. As before we conclude that $k_{1,1} + k_{2,2} \geq m - \upsilon \geq m - 1$. Therefore the Shi condition hold for this case as well.

We assume that our claim holds for all $\ell < n$ and we will show that it holds for n as well. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n^m$ and consider the partitions $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ and $\lambda^{(4)}$ defined in Lemma 4.5. Let $T^{(h)} = \{k_{i,j}^{(h)} \mid 1 \le i \le j \le n-1\} := \bar{\varphi}_{n-1}(\lambda^{(h)})$ for $h \in \{1,2,4\}$ and $T^{(3)} = \{k_{i,j}^{(3)} \mid 1 \le i \le j \le n-2\} := \bar{\varphi}_{n-2}(\lambda^{(3)})$. From the induction hypothesis, we have that $T^{(h)} \in \mathcal{R}_{n-1}^m$ for all $h \in \{1,2,4\}$ and $T^{(3)} \in \mathcal{R}_{n-2}^m$. Thus, from Equations (4.3), (4.4), (4.6) and the induction hypothesis we have that that the Shi condition (3.2) holds for each triplet $\{k_{i,j}, k_{i,l}, k_{l+1,j}\}$ with $(i,l,j) \neq (1, n-1, n)$.

Hence, it remains to show that the Shi condition (3.2) holds for the triplet $\{k_{1,n}, k_{1,n-1}, k_{n,n}\}$ as well. We distinguish two cases: (1) $k_{1,n} < m$ and (2) $k_{1,n} = m$.

(1) Let $k_{1,n} < m$. Lemma 4.3 (i) implies that $k_{1,n-1} < m$. In this case condition (3.2) reduces to showing that $k_{1,n} - k_{n,n} - k_{1,n-1} \in \{0,1\}$. By the definition of $k_{1,n}$, we have that

$$k_{1,n} = \frac{\lambda_1 + \sum_{l=2}^{n} k_{l,n} + \upsilon}{n}, \text{ where } \upsilon \in [n-1]_{\ge 0}. \text{ Moreover, } \lambda_1 = \sum_{\ell=1}^{n} k_{1,\ell}. \text{ We therefore have:}$$

$$nk_{1,n} = \lambda_1 + \sum_{l=2}^{n} k_{l,n} + \upsilon = \sum_{\ell=1}^{n} k_{1,\ell} + \sum_{l=2}^{n} k_{l,n} + \upsilon = k_{1,n} + \sum_{\ell=1}^{n-1} k_{1,\ell} + \sum_{\ell=2}^{n} k_{\ell,n} + \upsilon \leftrightarrow$$

$$(4.7) \qquad (n-1)k_{1,n} = \sum_{\ell=1}^{n-1} k_{1,\ell} + \sum_{\ell=2}^{n} k_{\ell,n} + \upsilon.$$

We recall now that $k_{1,n-1} < m$, thus for every $1 \le \ell \le n-2$ we have that $k_{1,n-1} = k_{1,\ell} + k_{\ell+1,n-1} + \delta_\ell$, where $\delta_\ell \in \{0,1\}$. Hence,

$$(4.8) \qquad \sum_{\ell=1}^{n-2} k_{1,n-1} = \sum_{\ell=1}^{n-2} k_{1,\ell} + \sum_{\ell=1}^{n-2} k_{\ell,n-1} + \sum_{\ell=1}^{n-2} \delta_\ell \Leftrightarrow$$

$$\sum_{\ell=1}^{n-2} k_{1,\ell} = \sum_{\ell=1}^{n-2} k_{1,n-1} - \sum_{\ell=1}^{n-2} k_{\ell,n-1} - \sum_{\ell=1}^{n-2} \delta_\ell \Leftrightarrow$$

$$(4.9) \qquad \sum_{\ell=1}^{n-2} k_{1,\ell} = (n-2)k_{1,n-1} - \sum_{\ell=1}^{n-2} k_{\ell,n-1} - \sum_{\ell=1}^{n-2} \delta_\ell.$$

On the other hand, for every $2 \le \ell \le n-1$, the entry $k_{\ell,n}$ satisfies the Shi condition, therefore, $k_{\ell,n} = k_{\ell,n-1} + k_{n,n} + \delta_\ell'$, where $\delta_\ell' \in \{0,1\}$. Thus,

$$\sum_{\ell=2}^{n-1} k_{\ell,n} = \sum_{\ell=2}^{n-1} k_{\ell,n-1} + \sum_{\ell=2}^{n-1} k_{n,n} + \sum_{\ell=2}^{n-1} \delta_\ell' \Leftrightarrow$$

$$(4.10) \qquad \sum_{\ell=2}^{n-1} k_{\ell,n} = \sum_{\ell=2}^{n-1} k_{\ell,n-1} + (n-2)k_{n,n} + \sum_{\ell=2}^{n-1} \delta_\ell'.$$

Substituting Equations (4.9) and (4.10) in Equation (4.7) we get:

$$(n-1)k_{1,n} = (n-1)(k_{1,n-1} + k_{n,n}) - \sum_{\ell=1}^{n-2} k_{\ell,n-1} - \sum_{\ell=1}^{n-2} \delta_\ell + \sum_{\ell=2}^{n-1} k_{\ell,n-1} + \sum_{\ell=2}^{n-1} \delta_\ell' + \upsilon.$$

Hence,

$$k_{1,n} = k_{1,n-1} + k_{n,n} + \kappa,$$

where $\kappa = \frac{\sum_{\ell=2}^{n-1} \delta_\ell' - \sum_{\ell=1}^{n-2} \delta_\ell + \upsilon}{n-1}$. Since $k_{1,n}, k_{1,2}, k_{2,n} \in \mathbb{N}$, it follows that $\kappa \in \mathbb{Z}$. Moreover, from the fact that $\delta_\ell, \delta_\ell' \in \{0,1\}$ and $\upsilon \in [n-1]_{\ge 0}$, one can easily see that $\kappa \in \{0,1\}$, which completes our claim for $k_{1,n} < m$.

(2) Let $k_{1,n} = m$. In this case, showing that the triplet $\{k_{1,n}, k_{1,n-1}, k_{n,n}\}$ satisfies the Shi condition, reduces to showing that $k_{1,n-1} + k_{n,n} \ge m-1$. Suppose that $k_{1,\ell} = m$ for some $1 \le \ell \le n-1$. Then, Lemma 4.3 implies that $k_{1,n-1} = m$ as well, thus the result follows. We

argue similarly if $k_{\ell,n} = m$ for some $2 \leq \ell \leq n$ (in this case we apply Lemma 4.6). Finally, suppose that $k_{1,j}, k_{i,n} < m$ for every $1 \leq i \leq n-1$ and $2 \leq j \leq n$. Then, the Shi conditions for each triplet are always as in the upper case of (3.2). We can therefore follow verbatim the computation done in the case where $k_{1,n} < m$. In particular, the definition of $k_{1,n}$ implies that

$k_{1,n} = m \leq \left\lceil \frac{\lambda_1 + \sum\limits_{l=2}^{n} k_{\ell,n}}{n} \right\rceil$ which, after the computations, gives $m = k_{1,n} \leq k_{1,n-1} + k_{n,n} + \kappa$,

where $\kappa \in \{0, 1\}$. Therefore, $\bar{\varphi}_n(\mathcal{P}_n^m) \subseteq \mathcal{R}_n^m$.

We will prove now that $\mathcal{R}_n^m \subseteq \bar{\varphi}_n(\mathcal{P}_n^m)$. Let $T = \{k_{i,j} \mid 1 \leq i \leq j \leq n\} \in \mathcal{R}_n^m$ and consider the partition λ whose parts are the sums of the entries of each row of T. Clearly then $\lambda \in \mathcal{P}_n^m$, and from the proof of Theorem 4.2 it follows that $T = \bar{\varphi}_n(\lambda)$, which implies that $T \in \bar{\varphi}_n(\mathcal{P}_n^m)$. This completes the proof of the lemma. $\square$

Proof of Theorem 4.1. The result follows immediately from Theorem 4.2, Corollary 4.4 and Lemma 4.7. $\square$

Proof of Theorem 1.3. From Theorem 4.1 we have that $\varphi_n : \mathcal{R}_n^m \to \mathcal{P}_n^m$ is a bijection. Finally, to show that φ_n has the property in the statement of the theorem, notice that $\lambda_i = m(n - i + 1)$ if and only if $k_{i,i} = k_{i,i+1} = \cdots = k_{i,n} = m$, which in turn implies that $H_{\alpha_i,m}$ is a separating wall. $\square$

5. FROM FACETS OF $\mathcal{D}_n^m$ TO PARTITIONS IN $\mathcal{P}_n^m$

In this section we prove Theorem 1.4 combinatorially. In particular, we give a bijection that characterizes the facets of $\mathcal{D}_n^m$ with respect to the negative simple roots they contain, in terms of partitions in P_n^m.

Definition 5.1. *Let P be a polygon with $m(n + 1) + 2$ vertices labeled by distinct integers of the set $[m(n + 1) + 1]_{\geq 0}$ and let d be an m-diagonal of P. The smallest label between the endpoints of d is called* initial *point of d.*

Consider now an $m(n+1) + 2$-gon P with its vertices labeled by the integers in $[m(n+1) + 1]_{\geq 0}$ in counterclockwise order. Let also D be a maximal m-dissection of P and let $\{t_1, t_2, \ldots, t_n\}$ be the multiset consisting of the initial points of the diagonals in D. Without loss of generality, we may assume that $t_1 \geq t_2 \geq \cdots \geq t_n$. One can show that $t_i \leq m(n - i + 1)$ for all $1 \leq i \leq n$, therefore, $(t_1, t_2, \ldots, t_n)$ is an element of $\mathcal{P}_n^m$. We consider the map $\psi_n' : \mathcal{D}_n^m \to \mathcal{P}_n^m$, which sends every dissection $D \in \mathcal{D}_n^m$ to the partition defined by the initial points of D, and note that it is a bijection. Even though the map ψ_n' describes an obvious way to associate integer partitions to polygon dissections, it can not be adopted in our case, since the composition $\varphi_n^{-1} \circ \psi_n'$ does not preserve the property of Theorem 1.1. Indeed, let D be the dissection of an $(m(n + 1) + 2)$-gon with diagonals $\{im, mn + 1\}$ where $1 \leq i \leq n$. These diagonals, which are all incident to $mn + 1$, do not form a "snake" when $n > 2$. Thus, the dissection D does not contain all the negative simple roots. Now, applying the above bijection we have $\psi_n'(D) = (mn, m(n - 1), \ldots, m)$. On the other hand, from Theorem 1.3, the partition $(mn, m(n - 1), \ldots, m)$ corresponds to the Shi tableau with all entries equal to m. This tableau represents the region having separating walls all hyperplanes of the form $H_{\alpha,m}$, where α is a simple root, which implies that the bijection $\varphi_n^{-1} \circ \psi_n'$ does not preserve the property stated in Theorem 1.1.

Although the map ψ_n' does not give us the required characterization, it constitutes the key-idea for the Proof of Theorem 1.4. Based on the bijection ψ_n', we relabel the vertices of P so that the property of Theorem 1.1 is preserved. The relabeling will give us an equivalent way to describe the negative simple roots, which will be consistent with the property we want to preserve. In the remainder of the paper, when we write that a vertex of a polygon P lies *on the right* (resp. *on the left*) of some other vertex of P, we mean on the right (resp. on the left) with respect to the center of the polygon.

Definition 5.2. *Let P be a polygon with $m(n + 1) + 2$ vertices labeled from 0 to $m(n + 1) + 1$ as follows: Fix a vertex labeled with 0. The vertices on the right of 0 are labeled with the numbers $k \in$*
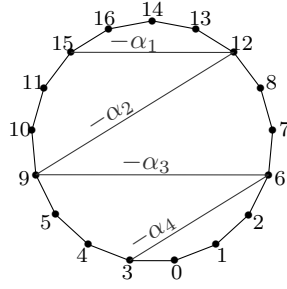
FIGURE 6. The alternating type-A labeling together with the negative simple roots, for $m = 3$ and $n = 4$.

$[m(m+1)+1]$ for which $\lfloor \frac{k}{m} \rfloor$ is even and so that the labels increase in the counterclockwise direction from vertex 0. Similarly, the vertices on the left of 0 are labeled with those $k \in [m(m+1)+1]$ for which $\lfloor \frac{k}{m} \rfloor$ is odd and so that the labels increase in the clockwise direction. We call this the alternating type-A labeling.

5.0.1. *Representation of negative simple roots for type A.* Let P be an $(m(n+1)+2)$-gon, fix some vertex 0 and consider its alternating type A-labeling. For each $1 \leq i \leq n$ we identify the negative simple root $-\alpha_i$ with the diagonal having endpoints $(n-i+1)m$ and $(n-i+2)m$. Notice that again the negative simple roots form a "snake", and thus the colored positive roots can be defined as in Section 2.2. For instance, for $m = 3$ and $n = 4$, the negative simple roots $-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4$ correspond to the diagonals $\{12, 15\}, \{9, 12\}, \{6, 9\}, \{3, 6\}$ respectively (see Figure 6).

Proof of Theorem 1.4. Consider an $(m(n+1)+2)$-gon P with alternating type-A labeling. Let $D \in \mathcal{D}_n^m$ be a dissection with diagonals $d_1, d_2, \ldots, d_n$. Let also $\{t_1, t_2, \ldots, t_n\}$ be the multiset consisting of all the initial points t_i. Without loss of generality we may assume that $t_1 \geq t_2 \geq \cdots \geq t_n$ and proceed as we did with ψ_n' (described in the beginning of this subsection). That is, we set $\psi_n(D) = (t_1, t_2, \ldots, t_n)$.

We first need to prove that ψ_n is well defined, or equivalently that $t_i \leq m(n-i+1)$ for all $1 \leq i \leq n$. Assume the contrary and let $i_0 \in [n]$ be the greatest index for which $t_{i_0} > m(n-i_0+1)$. Thus $t_i \leq m(n-i+1)$ for all $i_0 < i \leq n$. Since $m(n-i_0+1) < t_{i_0} \leq t_{i_0+1} \leq \cdots \leq t_n$, we deduce that the m-diagonals $d_1, d_2, \ldots, d_{i_0}$ should lie in the (mi_0+2)-gon defined by the diagonal corresponding to the root $-\alpha_{i_0}$ and the vertices $m(n-i_0+1), \ldots, m(n-i_0+2)$. But this is a contradiction, since an (mi_0+2)-gon cannot contain i_0 many m-diagonals. Thus the map ψ_n is well defined.

To see that ψ_n is a bijection we construct its inverse. We proceed by induction on n, the case $n = 1$ being trivial. Assume that we have constructed the bijection for $n-1$. Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an element of $\mathcal{P}_n^m$ and P be an $(m(n+1)+2)$-gon with the alternating type-A labeling. We will construct a dissection $D \in \mathcal{D}_n^m$ of P with set of initial points $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We consider the vertex of P which is labeled by λ_1. Among the two vertices which lie $(m+1)$-many vertices apart from λ_1, we denote by $\bar{\lambda}_1$ be the one with the greater label. Since $\lambda_1 \leq mn$, it follows from the definition of the alternating labeling that $\bar{\lambda}_1 \in \{mn+1, mn+2, \ldots, m(n+1)+1\}$, which implies that $\lambda_1 < \bar{\lambda}_1$. We set d_1 to be the diagonal of P with endpoints λ_1 and $\bar{\lambda}_1$. Clearly, d_1 dissects P into a $(m+2)$-gon and a $(mn+2)$-gon, which we denote by P_1. Note that P_1 contains all the vertices λ_i, for $2 \leq i \leq n$ and possibly some with greater labels. From the induction hypothesis, we can associate to the partition $(\lambda_2, \lambda_3, \ldots, \lambda_n)$ a dissection of the polygon P_1 having diagonals $d_1, d_2, \ldots, d_{n-1}$, where $d_i = \{\lambda_i < \bar{\lambda}_i\}$. We leave it to the reader to check that the map which sends λ to the dissection containing the diagonals $d_1, d_2, \ldots, d_n$ is indeed the inverse of ψ_n. By the representation of the negative simple roots as a "snake", it follows that a dissection contains the negative simple root $-\alpha_i$ if and only if it contains the diagonal $\{m(n-i+1), m(n-i+2)\}$. By induction on n one can show that this occurs if and only if the partition $\psi_n(D)$ has i-th part equal to $(n-i+1)m$. $\qquad\square$

6. Conclusion and ongoing work

We complete this paper with the proof of Theorem 1.1, which is a direct consequence of Theorem 1.3 and Theorem 1.4. We also present an application of the maps φ_n and ψ_n of Theorems 1.3 and 1.4. Finally, we briefly discuss on our ongoing work and state an open problem.

Theorem 1.1. We consider the map $\omega_n : \mathcal{D}_n^m \to \mathcal{R}_n^m$ with $\omega_n = \varphi_n^{-1} \circ \psi_n$. The result follows directly from Theorems 1.3 and 1.4 □

6.1. **Application.** Using the bijections provided in Theorems 1.3 and 1.4, we give a combinatorial proof of the fact that the number of facets of $\Delta_+^m(A_n)$ as well as the number of regions in $\mathrm{Cat}^m(A_n)$ is equal to $\frac{1}{n+1}\binom{m(n+1)+n-1}{n}$.

Corollary 6.1. $\#\mathcal{D}_n^m = \#\mathcal{R}_n^m = \frac{1}{n+1}\binom{m(n+1)+n-1}{n}$.

Proof. From [2, Lemma 4.1] a region in $\mathrm{Cat}^m(\Phi)$ is bounded if and only if it has no separating wall of type $H_{\alpha,m}$ where α is a simple root. Let α_i be a simple root of type A_n. In view of Theorem 1.3, the regions with no separating wall of type $H_{\alpha_i,m}$ biject to partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\mathcal{P}_n^m$ for which $\lambda_i < m(n-i+1)$. On the other hand, in view of Theorem 1.4, the facets not containing the root $-\alpha_i$, biject to the same partitions as well. These partitions can be viewed as paths from $(0,0)$ to (mn,n) which never touch the line $y = \frac{1}{m}x - 1$, or equivalently, as paths from $(0,0)$ to $(n, m(n+1))$ which never touch the line $y = mx$ after the point $(0,0)$. From [13, Theorem 3] we deduce that there are $\frac{1}{n+1}\binom{m(n+1)+n-1}{n}$ such paths. □

6.2. **Ongoing work.** In this paper we deal with the problem of finding a bijection between the set of dominant regions in the m-Catalan arrangement $\mathrm{Cat}^m(A_n)$ and that of facets of the m-generalized cluster complex $\Delta^m(A_n)$, where $m \in \mathbb{N}$. We further require the bijection to satisfy the property stated in Proposition 1.2. We answer this problem by providing a bijection which consists of two parts, where as intermediate step we use a certain set of integer partitions. Moreover, we use these integer partitions for enumerating the bounded regions of $\mathrm{Cat}^m(A_n)$ and the facets of $\Delta_+^m(A_n)$.

In [8] we focus on types B_n and C_n. So far we are able to characterize the set of facets of the generalized cluster complex $\Delta^m(B_n)$ and $\Delta^m(C_n)$ containing the negative simple root $-\alpha$, it terms of integers partitions. In particular we give a bijection between these sets of facets and the set $\mathcal{P}^m(B_n)$ of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for which $0 \leq \lambda_i \leq mn$. The construction of the bijections from the set of dominant regions of the arrangements $\mathrm{Cat}^m(B_n)$ and $\mathrm{Cat}^m(C_n)$ to the set $\mathcal{P}^m(B_n)$ is still in progress.

6.3. **Question.** Let Φ be a finite crystallographic root system. It would be very interesting to find a uniform bijection from the set $\mathcal{D}^m(\Phi)$ of facets of the m-generalized cluster complex, to the set $\mathcal{R}^m(\Phi)$ of dominant regions in the m-extended Catalan arrangement, which satisfies the property of Proposition 1.2.

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